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Quantumness of correlations and entanglement

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Generalized measurement schemes on one part of bipartite states, which would leave the set of *all* separable states insensitive are explored here to understand *quantumness of correlations* in a more general perspective. This is done by employing linear maps associated with generalized projective measurements. A generalized measurement corresponds to a quantum operation mapping a density matrix to another density matrix, preserving its positivity, hermiticity and traceclass. The Positive Operator Valued Measure (POVM) – employed earlier in the literature to optimize the measures of classical/quantum correlations – correspond to completely positive (CP) maps. The other class, the not completely positive (NCP) maps, are investigated here, in the context of measurements, for the first time. It is shown that such NCP projective maps provide a new clue to the understanding of quantumness of correlations in a general setting. Especially, the separability-classicality dichotomy gets resolved only when both the classes of projective maps (CP and NCP) are incorporated as optimizing measurements. An explicit example of a separable state – exhibiting non-zero quantum discord, when possible optimizing measurements are restricted to POVMs – is re-examined with this extended scheme incorporating NCP projective maps to elucidate the power of this approach.

Keywords: Correlations; projective maps; quantumness.

1. Introduction

Entanglement between subsystems of a composite state brought forth perplexing distinctions¹ between classical and quantum correlations. Fundamental significance of such incompatibility was highlighted by Bell's novel work². Following Werner³, it is believed that the statistical correlations between parts of a convex mixture of product (separable) states can be reproduced by a classical hidden

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variable model and they satisfy all Bell inequalities. The physical source of separable correlations being a classical preparation device, they are termed classical. In other words, quantum correlation owes its origin to the impossibility of expressing a composite quantum state as a convex combination of product states. However, several other measures of non-classical correlations – which are more general than entanglement – are drawing significant attention during the past few years^{4,5,6,7,8,9,10,11,12,13,14,15,16}. It is identified that non-classical correlations, other than entanglement, offer quantum advantage in some information processing tasks^{17,18}.

Now we proceed to elaborate on the concept of quantumness of correlations – other than that implied by entanglement. In classical probability theory, two random variables A and B are said to be correlated if their probability distribution, $P(a, b)$ cannot be expressed as a mere product of the marginal probabilities $P(a)$ and $P(b)$. Shannon mutual information

$$H(A : B) = H(A) + H(B) - H(A, B) \quad (1)$$

(where $H(A, B) = -\sum_{a,b} p(a, b) \log p(a, b)$, $H(A) = -\sum_a p(a) \log p(a)$, $H(B) = -\sum_b p(b) \log p(b)$) is an unequivocal measure of classical correlations.

In the quantum description, probability distributions are replaced by density operators and a bipartite density matrix $\hat{\rho}_{AB}$ is *correlated* if it cannot be expressed in a simple product form of its constituent subsystem density matrices $\hat{\rho}_A$, $\hat{\rho}_B$. The von Neumann mutual information,

$$\begin{aligned} S(A : B) &= S(\hat{\rho}_{AB} || \hat{\rho}_A \otimes \hat{\rho}_B) \\ &= S(\hat{\rho}_A) + S(\hat{\rho}_B) - S(\hat{\rho}_{AB}) \end{aligned} \quad (2)$$

(where $S(\hat{\rho}) = -\text{Tr}[\hat{\rho} \log \hat{\rho}]$) quantifies the *total* correlations – classical as well as quantum – in a bipartite state $\hat{\rho}_{AB}$. Distinguishing these two kinds of correlations gains basic importance – that too when one addresses the issue from a significantly different perspective – keeping aside the established separability-entanglement demarcation of correlations. It is with this view that Ollivier and Zurek (OZ)⁴ pointed towards characterizing quantumness of correlations in a bipartite system based on measurement perspective. They considered the quantum analogue of mutual information, which is sensitive to measurement on one part of the composite system as,

$$S(A : B) = S(\hat{\rho}_B) - \sum_{\alpha} p_{\alpha} S(\hat{\rho}_{B|A_{\alpha}}) \quad (3)$$

where

$$\begin{aligned} \hat{\rho}_{B|A_{\alpha}} &= \frac{\hat{\Pi}_{\alpha}^A \otimes I_B \hat{\rho}_{AB} \hat{\Pi}_{\alpha}^A \otimes I_B}{p_{\alpha}} \\ &= \hat{\Pi}_{\alpha}^A \otimes \hat{\rho}_{\alpha}^B \end{aligned} \quad (4)$$

denotes the conditional density operator, which results after a projective measurement $\{\hat{\Pi}_\alpha^A\}$ on subsystem A of the composite state $\hat{\rho}_{AB}$; $p_\alpha = \text{Tr}[\hat{\Pi}_\alpha^A \otimes I_B \hat{\rho}_{AB}]$ denotes the probability of outcome and $\hat{\rho}_\alpha^B = \text{Tr}_A[\hat{\rho}_{AB} |_{A_\alpha}]$.

OZ proposed *quantum discord* as the *minimum* difference between the two equivalent quantum analogs (2) and (3) of mutual information to characterize quantumness of correlations in $\hat{\rho}_{AB}$:

$$\delta(A, B)_{\{\hat{\Pi}_\alpha^A\}} = S(A : B) - \max_{\{\hat{\Pi}_\alpha^A\}} S(A : B) \quad (5)$$

where the maximization is done over complete, orthogonal projective measurements $\{\hat{\Pi}_\alpha^A\}$ on subsystem A .

A classically correlated bipartite state remains insensitive to a specific choice of projective measurement $\{\hat{\Pi}_\alpha^A\}$ on a part of the system – leading to vanishing quantum discord:

$$\begin{aligned} \delta(A, B)_{\{\hat{\Pi}_\alpha^A\}} = 0 \Rightarrow \\ \hat{\rho}_{AB}^{(\text{cl})} = \sum_\alpha \hat{\Pi}_\alpha^A \otimes I_B \hat{\rho}_{AB}^{(\text{cl})} \hat{\Pi}_\alpha^A \otimes I_B. \end{aligned} \quad (6)$$

Non-zero values of quantum discord quantify *quantumness* of correlations.

Expressing the classically correlated state in the basis $\{|\alpha\rangle\}$ of the orthogonal projectors, it is easy to see that

$$\begin{aligned} \hat{\rho}_{AB}^{(\text{cl})} &= \sum_{\alpha, \beta', \beta} \langle \alpha; \beta' | \hat{\rho}_{AB}^{(\text{cl})} | \alpha; \beta \rangle \hat{\Pi}_\alpha^A \otimes |\beta'\rangle\langle\beta| \\ &= \sum_\alpha q_\alpha \hat{\Pi}_\alpha^A \otimes \hat{\tau}_\alpha^B \end{aligned} \quad (7)$$

where $q_\alpha = \sum_\beta \langle \alpha; \beta | \hat{\rho}_{AB}^{(\text{cl})} | \alpha; \beta \rangle = \text{Tr}[\hat{\rho}_{AB}^{(\text{cl})}]$ and $\hat{\tau}_\alpha^B = \sum_{\beta, \beta'} \frac{\langle \alpha; \beta' | \hat{\rho}_{AB}^{(\text{cl})} | \alpha; \beta \rangle}{q_\alpha} |\beta'\rangle\langle\beta|$.

Clearly, the classically correlated states form a subset of separable states of the form $\left\{ \sum_\alpha q_\alpha \hat{\Pi}_\alpha^A \otimes \hat{\tau}_\alpha^B \right\}$. Quantum discord does not necessarily vanish for *all* separable states. In other words, it suggests that the concept of quantum correlations is more general than entanglement – as separable states too exhibit quantumness of correlations (non-zero quantum discord). OZ, however, based their discussion on quantumness of correlations by confining their attention only to orthogonal projective measurements $\{\hat{\Pi}_\alpha^A \otimes I_B\}$.

We give here an example of a two qubit separable state, which has non-zero quantum discord^{5,9}:

$$\begin{aligned} \hat{\rho}_{AB} &= p |0_A, 0_B\rangle \langle 0_A, 0_B| + (1-p) |+_A, +_B\rangle \langle +_A, +_B|, \\ 0 \leq p \leq 1, \quad |\pm\rangle &= \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle). \end{aligned} \quad (8)$$

Following a similar approach Henderson and Vedral (HV)⁵ independently investigated how to separate classical and quantum correlations. They employed general

positive operator valued measures (POVMs) to quantify classical correlations in the state $\hat{\rho}_{AB}$ in terms of the residual information entropy of B as follows:

$$C_A(\hat{\rho}_{AB}) = \max_{\{V_i^A\}} S(\hat{\rho}_B) - \sum_i q_i S(\hat{\rho}_i^B) \quad (9)$$

where $\hat{\rho}_i^B = \text{Tr}_A[V_i^A \otimes I_B \hat{\rho}_{AB} V_i^{A\dagger} \otimes I_B]/q_i$ is the density matrix of subsystem B after the measurement $\{V_i^A \otimes I_B\}$ is performed on A and $q_i = \text{Tr}_{AB}[V_i^A \otimes I_B \hat{\rho}_{AB} V_i^{A\dagger} \otimes I_B]$ denotes the probability of outcome. In a classically correlated state the residual information entropy of B *does not* increase under an optimal measurement scheme on A .

By analyzing some examples HV found that classical and entangled correlations do not add up to give total correlations⁵ i.e., $C_A(\hat{\rho}_{AB}) + E_{\text{RE}}(\hat{\rho}_{AB}) \neq S(A : B)$, where $E_{\text{RE}}(\hat{\rho}_{AB})$ denotes the relative entropy of entanglement. Hamieh et. al.⁹ showed that optimization of classical correlations⁵ in two qubit states may be achieved using orthogonal projective measurements themselves. This also leads to the identification that the classical correlations⁵ and the quantum discord⁴ add up to give the mutual information entropy in two-qubit states.

Another measure of quantum correlations is the one-way information deficit⁷ which is defined as the minimal increase of entropy after a projective measurement $\{\hat{\Pi}_\alpha^A\}$ on subsystem A is done:

$$\Delta^\rightarrow(\hat{\rho}_{AB}) = \min_{\{\hat{\Pi}_\alpha^A\}} S\left(\sum_\alpha \hat{\Pi}_\alpha^A \hat{\rho}_{AB} \hat{\Pi}_\alpha^A\right) - S(\hat{\rho}_{AB}). \quad (10)$$

The one-way information deficit vanishes only on states with zero quantum discord.

Quantum discord $\delta(A, B)_{\{\hat{\Pi}_\alpha^A\}}$, the HV classical correlations $C_A(\hat{\rho}_{AB})$ and the one-way information deficit $\Delta^\rightarrow(\hat{\rho}_{AB})$ are all asymmetric with respect to measurements on the subsystems A and B . Quantum deficit – one other measure of non-classical correlations – which is symmetric about the subsystems A, B , was proposed by Rajagopal and Rendell⁶ as follows:

$$D_{AB} = S(\hat{\rho}_{AB} || \hat{\rho}_{AB}^{(d)}) = \text{Tr}[\hat{\rho}_{AB} \log \hat{\rho}_{AB}] - \text{Tr}[\hat{\rho}_{AB} \log \hat{\rho}_{AB}^d], \quad (11)$$

where $\hat{\rho}_{AB}^{(d)} = \sum_{a,b} P(a, b) \hat{\Pi}_a^A \otimes \hat{\Pi}_b^B$, where $\hat{\Pi}_a^A, \hat{\Pi}_b^B$ correspond to eigenprojectors of the subsystems $\hat{\rho}_A, \hat{\rho}_B$ with $P(a, b) = \langle a, b | \hat{\rho}_{AB} | a, b \rangle$ denoting the diagonal elements of $\hat{\rho}_{AB}$, in its subsystem eigen basis and $P(a) = \sum_b P(a, b)$, $P(b) = \sum_a P(a, b)$ the eigenvalues of $\hat{\rho}_A, \hat{\rho}_B$ respectively. The quantum deficit D_{AB} determines the quantum excess of correlations in the state $\hat{\rho}_{AB}$, with reference to its classically decohered counterpart $\hat{\rho}_{AB}^{(d)}$ and it vanishes *iff* $\hat{\rho}_{AB} \equiv \hat{\rho}_{AB}^{(d)}$. It may be noted that bipartite states with zero quantum deficit have vanishing quantum discord. Another important feature is that evaluating quantum deficit is easier compared to the other measures of correlations outlined above, as no optimization procedure is involved in its evaluation.

It appears natural to raise the question¹²: are there more general measurement schemes on one part of bipartite states, which would leave *all* the separable

states insensitive? Possibility of such generalized measurements would resolve the dichotomy of separability vs classicality of correlations. Furthermore any measure of quantumness of correlations could then be identified with that of entanglement itself. In this paper, we analyze the basic features of generalized measurement scheme which could imply that absence of entanglement and classicality are synonymous. We show that not completely positive (NCP) projective maps – in contrast to POVMs – are the essential ingredients of generalized measurements on one end of a bipartite system that leave separable states unaltered.

2. A generalized measure of quantumness of correlations

We discuss some specific properties of quantum discord so as to extend the notion of *quantumness of correlations* in a bipartite system by invoking generalized measurements.

Consider a bipartite state $\hat{\rho}_{AB}$, for which optimization of quantum discord $\delta(A, B)_{\{\hat{\Pi}_\alpha^A\}}$ is realized in terms of a complete orthogonal projective set $\{\hat{\Pi}_\alpha^A\}$. The state left after measurement is given by,

$$\begin{aligned}\hat{\rho}_{AB}^{\mathcal{D}} &= \sum_{\alpha} \hat{\Pi}_\alpha^A \otimes I_B \hat{\rho}_{AB} \hat{\Pi}_\alpha^A \otimes I_B \\ &= \sum_{\alpha} p_{\alpha} \hat{\rho}_{B|A_{\alpha}}\end{aligned}\quad (12)$$

where $\hat{\rho}_{B|A_{\alpha}}$ is the conditional density operator (see (4)) and $p_{\alpha} = \text{Tr}[\hat{\Pi}_\alpha^A \otimes I_B \hat{\rho}_{AB}]$. Using the property ⁴

$$S(\hat{\rho}_{AB}^{\mathcal{D}}) = - \sum_{\alpha} p_{\alpha} \log p_{\alpha} + \sum_{\alpha} p_{\alpha} S(\hat{\rho}_{B|A_{\alpha}}) \quad (13)$$

one can express quantum discord (see (5)) in terms of the relative entropies as follows:

$$\begin{aligned}\delta(A, B)_{\{\hat{\Pi}_\alpha^A\}} &= S(\hat{\rho}_{AB}^{\mathcal{D}}) - S(\hat{\rho}_{AB}) + S(\hat{\rho}_A) - \sum_{\alpha} p_{\alpha} \log p_{\alpha} \\ &= S(\hat{\rho}_{AB} || \hat{\rho}_{AB}^{\mathcal{D}}) + S(\hat{\rho}_A || \hat{\rho}_A^{\mathcal{D}})\end{aligned}\quad (14)$$

This structure of quantum discord clearly projects out the fact that (i) $\delta(A, B)_{\{\hat{\Pi}_\alpha^A\}} \geq 0$ as the relative entropies $S(\hat{\rho}_{AB} || \hat{\rho}_{AB}^{\mathcal{D}})$, $S(\hat{\rho}_A || \hat{\rho}_A^{\mathcal{D}})$ are positive semidefinite quantities (ii) they vanish iff $\hat{\rho}_{AB} \equiv \hat{\rho}_{AB}^{\mathcal{D}}$ i.e., if the state $\hat{\rho}_{AB}$ remains insensitive to projective measurement $\{\hat{\Pi}_\alpha^A\}$. Moreover, observing that the state after measurement is a classically correlated state i.e., $\hat{\rho}_{AB}^{\mathcal{D}} = \sum_{\alpha} p_{\alpha} \hat{\Pi}_\alpha^A \otimes \rho_{\alpha}^B$, the quantum discord gets related to ¹⁴ *distance between the given state $\hat{\rho}_{AB}$ and its closest classically correlated state $\hat{\rho}_{AB}^{\mathcal{D}}$* (where *distance* is considered in terms of the relative entropy).

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A natural extension of the measure of quantumness of correlations¹² – as a distance between the given bipartite state and the closest state realized after measurements at one end of the state – will be outlined in the following.

Let us consider the set of all tripartite density operators $\{\hat{\rho}_{A'AB}\}$ in an extended Hilbert space $\mathcal{H}_{A'} \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, such that the bipartite state $\hat{\rho}_{AB}$ under investigation is a marginal of this extended system:

$$\text{Tr}_{A'}[\hat{\rho}_{A'AB}] = \hat{\rho}_{AB}. \quad (15)$$

Now, carrying out an orthogonal projective measurement $\Pi_i^{(A'A)}$, $i = 1, 2, \dots$, on one of the subsystems $A'A$ of the tripartite state $\hat{\rho}_{A'AB}$ we obtain,

$$\begin{aligned} \hat{\rho}_{A'AB} &\rightarrow \hat{\rho}_{A'AB}^{(i)} = \frac{1}{p_i} \left[\hat{\Pi}_i^{(A'A)} \otimes I_B \hat{\rho}_{A'AB} \hat{\Pi}_i^{(A'A)} \otimes I_B \right] \\ \text{and } \hat{\rho}_{AB} &\rightarrow \hat{\rho}_{AB}^{(i)} = \frac{1}{p_i} \text{Tr}_{A'} \left[\hat{\Pi}_i^{(A'A)} \otimes I_B \hat{\rho}_{A'AB} \hat{\Pi}_i^{(A'A)} \otimes I_B \right] \end{aligned} \quad (16)$$

where $p_i = \text{Tr}_{A'AB} [\hat{\Pi}_i^{(A'A)} \otimes I_B \hat{\rho}_{A'AB}]$ denotes the probability of occurrence of i^{th} outcome.

We define *Quantumness* \mathcal{Q}_{AB} associated with a bipartite state $\hat{\rho}_{AB}$ as the relative entropy

$$\mathcal{Q}_{AB} = \min_{\{\hat{\Pi}_i^{(A'A)}, \hat{\rho}_{A'AB}\}} S(\hat{\rho}_{AB} || \hat{\rho}_{AB}^{\mathcal{R}}) \quad (17)$$

Here, $\hat{\rho}_{AB}^{\mathcal{R}} = \text{Tr}_{A'} [\sum_i \hat{\Pi}_i^{(A'A)} \otimes I_B \hat{\rho}_{A'AB} \hat{\Pi}_i^{(A'A)} \otimes I_B]$, denotes the residual state of the bipartite system, left after the generalized projective measurement is performed. The minimum in Eq. (17) is taken over the set $\{\hat{\Pi}_i^{(A'A)}\}$ of projectors on the subsystems $A'A$ of *all* possible extended states $\{\hat{\rho}_{A'AB}\}$, which contain the given bipartite state $\hat{\rho}_{AB}$ as their marginal system.

The quantumness, $\mathcal{Q}_{AB} \geq 0$ (by definition), for all generalized measurements - the equality sign holding iff $\hat{\rho}_{AB}^{\mathcal{R}} = \hat{\rho}_{AB}$ i.e., quantumness vanishes iff the bipartite state $\hat{\rho}_{AB}$ remains insensitive to generalized measurement $\{\hat{\Pi}_i^{(A'A)}\}$.

Corresponding to a chosen measurement scheme $\{\hat{\Pi}_i^{(A'A)}\}$ we may express the extended state $\hat{\rho}_{A'AB}$ in terms of the complete, orthogonal set of basis states $\{|i\rangle_{A'A} \otimes |\beta\rangle_B\}$ as,

$$\hat{\rho}_{A'AB} = \sum_{i', i, \beta', \beta} P(i', \beta'; i, \beta) |i'\rangle_{A'A} \langle i'| \otimes |\beta'\rangle_B \langle \beta|.$$

We then obtain,

$$\hat{\rho}_{A'AB}^{\mathcal{R}} = \hat{\Pi}_i^{(A'A)} \otimes I_B \hat{\rho}_{A'AB} \hat{\Pi}_i^{(A'A)} \otimes I_B = \sum_{\beta', \beta} P(i, \beta'; i, \beta) \hat{\Pi}_i^{(A'A)} \otimes |\beta'\rangle_B \langle \beta| \quad (18)$$

which leads in turn to

$$\begin{aligned}\hat{\rho}_{AB}^{\mathcal{R}} &= \text{Tr}_{A'} \left[\sum_i \hat{\Pi}_i^{(A'A)} \otimes I_B \hat{\rho}_{A'AB} \hat{\Pi}_i^{(A'A)} \otimes I_B \right] \\ &= \sum_i p_i \hat{\rho}_i^A \otimes \hat{\rho}_i^B\end{aligned}\quad (19)$$

where $\hat{\rho}_i^A = \text{Tr}_{A'} [\hat{\Pi}_i^{(A'A)}]$ and

$$\begin{aligned}\hat{\rho}_i^B &= \sum_{\beta', \beta} \frac{P(i, \beta'; i, \beta)}{p_i} |\beta'\rangle_B \langle \beta|, \\ p_i &= \text{Tr} [\hat{\Pi}_i^{(A'A)} \otimes I_B \hat{\rho}_{A'AB}] = \sum_{\beta} P(i, \beta; i, \beta).\end{aligned}$$

Clearly, the state $\hat{\rho}_{AB}^{\mathcal{R}}$ of the bipartite system – left after performing the generalized measurement $\{\hat{\Pi}_i^{(A'A)}\}$ on the part $A'A$ of the global system – is a separable state. As the optimization of quantumness \mathcal{Q}_{AB} is done over the set of all projectors $\{\hat{\Pi}_i^{(A'A)}\}$, and the set of all extended states $\{\hat{\rho}_{A'AB}\}$, it is readily seen that $\{\hat{\rho}_{AB}^{\mathcal{R}} = \hat{\varrho}_{AB}^{(\text{sep})}; \hat{\rho}_B = \text{Tr}[\hat{\rho}_{AB}^{\mathcal{R}}]\}$ corresponds to the set of all separable states which share the same subsystem density matrix $\hat{\rho}_B$ for the part B (i.e., the subsystem, which does not come under the direct action of generalized measurements $\{\hat{\Pi}_i^{(A'A)}\}$). We thus obtain

$$\begin{aligned}\mathcal{Q}_{AB} &= \min_{\{\hat{\Pi}_i^{(A'A)}, \hat{\rho}_{A'AB}\}} S(\hat{\rho}_{AB} || \hat{\rho}_{AB}^{\mathcal{R}}) \\ &= \min_{\{\hat{\varrho}_{AB}^{(\text{sep})}\}} S(\hat{\rho}_{AB} || \hat{\varrho}_{AB}^{(\text{sep})})\end{aligned}\quad (20)$$

with minimization taken over the set of all separable states $\{\hat{\varrho}_{AB}^{(\text{sep})}; \hat{\rho}_B = \text{Tr}[\hat{\varrho}_{AB}^{(\text{sep})}]\}$.

In other words, the generalized measure \mathcal{Q}_{AB} of quantumness of correlations corresponds to the distance between the given state $\hat{\rho}_{AB}$ with the closest separable state $\hat{\varrho}_{AB}^{(\text{sep})}$; $\text{Tr}[\hat{\varrho}_{AB}^{(\text{sep})}] = \hat{\rho}_B$. From Eq. (20) it is evident that quantumness \mathcal{Q}_{AB} is necessarily non-zero for all entangled bipartite states $\hat{\rho}_{AB}$ and vanishes for *all* separable states. Moreover, \mathcal{Q}_{AB} also serves as an upper bound to the relative entropy of entanglement¹⁹. While the evaluation of \mathcal{Q}_{AB} is as hard a task as that of relative entropy of entanglement, the significant point here is that it brings out the required generalized scheme of measurements, which resolve the dichotomy between *quantumness of correlations* and *entanglement*. Further, the established connection – viz., the quantumness of correlations is the distance between the given bipartite state with its closest separable state (sharing the same marginal state for the subsystem B) – highlights the merger of quantumness of correlations with quantum entanglement itself. This in turn ensures that any other *operational* measure of bipartite entanglement would faithfully reflect quantumness of correlations in the state.

We illustrate the scheme of generalized projective measurements on $A'A$ subsystem of an extended tripartite state $\hat{\rho}_{A'AB}$ of the separable state (8). An extended

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three qubit state

$$\hat{\rho}_{A'AB} = p |1_{A'}, 0_A, 0_B\rangle \langle 1_{A'}, 0_A, 0_B| + (1-p) |0_{A'}, +_A, +_B\rangle \langle 0_{A'}, +_A, +_B| \quad (21)$$

leads to the given two qubit state (8) by tracing over the A' qubit. We find that the complete, orthogonal set of projectors $\{\hat{\Pi}_i^{(A'A)}\}$ on $A'A$ constituted by

$$\begin{aligned} \hat{\Pi}_1^{(A'A)} &= |0_{A'}, +_A\rangle \langle 0_{A'}, +_A|, \\ \hat{\Pi}_2^{(A'A)} &= |0_{A'}, -_A\rangle \langle 0_{A'}, -_A|, \\ \hat{\Pi}_3^{(A'A)} &= |1_{A'}, 0_A\rangle \langle 1_{A'}, 0_A|, \\ \hat{\Pi}_4^{(A'A)} &= |1_{A'}, 1_A\rangle \langle 1_{A'}, 1_A| \end{aligned} \quad (22)$$

leaves the overall state (8) unaltered:

$$\hat{\rho}_{A'AB}^{\mathcal{R}} = \sum_{i=1}^4 \hat{\Pi}_i^{(A'A)} \otimes I_B \hat{\rho}_{A'AB} \hat{\Pi}_i^{(A'A)} \otimes I_B \equiv \hat{\rho}_{A'AB}. \quad (23)$$

So, we identify that the bipartite state (8) is insensitive under the generalized projective measurements (22)) i.e., $\hat{\rho}_{AB}^{\mathcal{R}} = \hat{\rho}_{AB}$ implying that $\mathcal{Q}_{AB} = S(\hat{\rho}_{AB} || \hat{\rho}_{AB}^{\mathcal{R}}) = 0$ in this state.

The generalized projective measurements on $A'A$ part of the extended state may be viewed as quantum maps, which transform density matrices $\hat{\rho}_A$ (before measurement) to density matrices $\hat{\rho}_A^{\mathcal{R}}$ (after measurement) – preserving their hermiticity, positivity and trace class. In the next section we investigate the properties of the linear map associated with the generalized measurements.

3. Linear \mathcal{A}, \mathcal{B} maps associated with generalized projective measurements

Dynamical \mathcal{A} and \mathcal{B} maps have been employed extensively by Sudarshan and co-workers to investigate open system evolution of quantum systems^{20,21,22}. Here, we elucidate the projective measurements $\{\hat{\Pi}_i^{(A'A)}\}$ on $\hat{\rho}_{A'A}$ in terms of linear \mathcal{A}, \mathcal{B} quantum maps on $\hat{\rho}_A$ – transforming it to the resultant density matrix $\hat{\rho}_A^{\mathcal{R}}$ – preserving the positivity, hermiticity and unit trace conditions. The elements $(\hat{\rho}_A^{\mathcal{R}})_{a_k a_l}$ after measurement are explicitly expressed in terms of those of initial density matrix $(\hat{\rho}_A)_{a_i a_j}$ via the \mathcal{A} map as^{20,21}

$$(\hat{\rho}_A^{\mathcal{R}})_{a_i a_j} = \sum_{a_k, a_l} \mathcal{A}_{a_i a_j; a_k a_l} (\hat{\rho}_A)_{a_k a_l}. \quad (24)$$

That the resultant density matrix $\hat{\rho}_A^{\mathcal{R}}$ is Hermitian and has unit trace leads to the conditions

$$\text{Hermiticity : } \mathcal{A}_{a_i a_j; a_k a_l} = \mathcal{A}_{a_j a_i; a_l a_k}^*, \quad (25)$$

$$\text{Trace preservation : } \sum_{a_i} \mathcal{A}_{a_i a_i; a_k a_l} = \delta_{a_k, a_l}, \quad (26)$$

In order to bring out the properties (25),(26) in a lucid manner, a realigned matrix \mathcal{B} is defined as:

$$\mathcal{B}_{a_i a_k; a_j a_l} = \mathcal{A}_{a_i a_j; a_k a_l}. \quad (27)$$

The hermiticity property (25) leads to the condition $\mathcal{B}_{a_i a_k; a_j a_l} = \mathcal{B}_{a_j a_l; a_i a_k}^*$, i.e., the map \mathcal{B} is hermitian.

In terms of the spectral decomposition $\mathcal{B}_{a_i a_k; a_j a_l} = \sum_{\alpha} \lambda_{\alpha} M_{a_i a_k}^{(\alpha)} M_{a_j a_l}^{(\alpha)*}$, the action of the \mathcal{B} map on the density matrix is then readily identified as,

$$\begin{aligned} (\hat{\rho}_A^{\mathcal{R}})_{a_i a_j} &= \sum_{\alpha, a_k, a_l} \lambda_{\alpha} M_{a_i a_k}^{(\alpha)} M_{a_j a_l}^{(\alpha)*} (\hat{\rho}_A)_{a_k a_l} \\ \Rightarrow \hat{\rho}_A^{\mathcal{R}} &= \sum_{\alpha} \lambda_{\alpha} M^{(\alpha)} \hat{\rho}_A M^{(\alpha)\dagger} \end{aligned} \quad (28)$$

and this corresponds to POVM on $\hat{\rho}_A$ provided $\lambda_{\alpha} \geq 0$ or a completely positive (CP) map associated with projective measurement; otherwise it is a not completely positive (NCP) map.

We focus on finding the CP/NCP nature of the projective quantum map transforming the single qubit state $\hat{\rho}_A = \text{Tr}_{A'B}[\hat{\rho}_{A'AB}]$ (before measurement) with $\hat{\rho}_A^{\mathcal{R}} = \text{Tr}_{A'B} \left[\left\{ \sum_i \Pi_i^{(A'A)} \hat{\rho}_{A'AB} \Pi_i^{(A'A)} \right\} \right]$ (after measurement) – corresponding to the specific measurement scheme $\{\Pi_i^{(A'A)}\}$ (see Eq. (22)) on the state $\hat{\rho}_{A'AA}$ of Eq. (21) – i.e., in the specific example discussed in Sec. 2. It is pertinent to point out here that the state $\hat{\rho}_{A'AB}$, and hence the reduced state $\hat{\rho}_A$, remain insensitive to the projective measurement (22), as has already been illustrated explicitly in Sec. 2 (see Eqs.(21-23)). The corresponding quantum map transforming $\hat{\rho}_A \rightarrow \hat{\rho}_A^{\mathcal{R}}$ must reveal this insensitivity.

In order to deduce the explicit structure of the projective A, B maps, we employ the concept of *assignment map*²³. Explicit technical details and derivations are elaborated in Appendix. We obtain the \mathcal{B} map (see Appendix Eq. (52)) associated with this particular example as,

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \quad (29)$$

where the rows and columns are labeled as $\{00, 01, 10, 11\}$. The associated \mathcal{A} matrix is then obtained as (using (27)),

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (30)$$

- Applying the measurement map \mathcal{A} of Eq. (30) on the state $\hat{\rho}_A = \text{Tr}_{A'B}[\hat{\rho}_{A'AB}] = p|0_A\rangle\langle 0_A| + (1-p)|+_A\rangle\langle +_A|$ (the state before measurement) it may be seen explicitly (following Eq. (24)) that $(\hat{\rho}_A^{\mathcal{R}})_{a_i a_j} = \sum_{a_k, a_l=0,1} \mathcal{A}_{a_i a_j; a_k a_l} (\hat{\rho}_A)_{a_k a_l} \equiv [\hat{\rho}_A]_{a_i a_j}$ i.e., the state is insensitive to this measurement. It may be recalled here that the projective measurement (22) leaves the tripartite state (21) – and hence its subsystems $\hat{\rho}_{AB}$ (and also $\hat{\rho}_A$) – undisturbed as is illustrated in Sec. 2. This in turn led to the implication that the quantumness of correlation \mathcal{Q}_{AB} vanishes for the separable state $\hat{\rho}_{AB}$ of Eq. (8) – whereas, quantum discord and quantum deficit are non-zero.
- The eigenvalues of \mathcal{B} are readily found to be $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2})$, implying that the projective measurement (22) on the state $\hat{\rho}_A$ corresponds to a NCP map.

In other words, we reach a crucial identification that the map, which leaves the state $\hat{\rho}_{AB}$ of Eq. (8) insensitive under measurements is NCP. Our generalized measure of quantumness (17) may also be expressed as,

$$\mathcal{Q}_{AB} = \min_{\{\text{CP/NCP projective maps on } A\}} S(\hat{\rho}_{AB} || \hat{\rho}_{AB}^{\mathcal{R}}) \quad (31)$$

where we emphasize that positivity, hermiticity and trace of the given density matrix are preserved by the optimizing CP/NCP projective maps. A comparison of Eq. (31) with the alternate form (given in Eq. (20)), suggests that both the classes of projective maps (CP and NCP) need to be incorporated in order to deem quantumness of correlations as synonymous with quantum entanglement itself. Having thus established that the quantumness of correlations \mathcal{Q}_{AB} of bipartite states is non-zero only for entangled states, we point out once again that any other *operational* measure of entanglement would necessarily imply such non-classicality of correlations – and this identification takes away the burden of evaluating \mathcal{Q}_{AB} (where the optimization procedure turns out to be a demanding task) per se to infer quantumness.

4. Summary

Sudarshan and coworkers^{20,21} put forward the conceptual formulation of quantum theory of open system evolution in terms of dynamical \mathcal{A}, \mathcal{B} maps almost 50 years ago and they also investigated it in the more general setting^{22,24,25} – including NCP dynamical maps. In this paper we highlight the important role of NCP projective maps in the context of measurements. It is shown that incorporating generalized measurement schemes – including both CP as well as NCP maps – resolves the dichotomy of separability vs classicality of correlations.

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Appendix: \mathcal{A} , \mathcal{B} maps associated with projective measurement

Let us consider complete, orthonormal set of projective measurements $\{\Pi_i^{(A'A)}\}$ on $\hat{\rho}_{A'A}$. We proceed to construct the \mathcal{A}, \mathcal{B} maps transforming the system state $\hat{\rho}_A = \text{Tr}_{A'}[\hat{\rho}_{A'A}]$ (before measurement) to the state $\hat{\rho}_A^{\mathcal{R}} = \text{Tr}_{A'}[\sum_i \hat{\Pi}_i^{A'A} \hat{\rho}_{A'A} \hat{\Pi}_i^{A'A}]$ (after measurement).

$$\begin{aligned} \hat{\rho}_A^{\mathcal{R}} &= \text{Tr}_{A'} \left[\sum_i \hat{\Pi}_i^{A'A} \hat{\rho}_{A'A} \hat{\Pi}_i^{A'A} \right] \\ &= \sum_i \text{Tr}_{A'A}[\hat{\Pi}_i^{A'A} \hat{\rho}_{A'A}] \text{Tr}_{A'}[\hat{\Pi}_i^{A'A}] \\ &= \sum_i \mathcal{P}_i \hat{\rho}_i^A \end{aligned} \quad (32)$$

where we have denoted

$$\text{Tr}_{A'}[\hat{\Pi}_i^{A'A}] = \hat{\rho}_i^A \quad (33)$$

$$\mathcal{P}_i = \text{Tr}_{A'A}[\hat{\Pi}_i^{A'A} \hat{\rho}_{A'A}] \quad (34)$$

We simplify $\mathcal{P}_i = \text{Tr}_{A'A}[\hat{\Pi}_i^{A'A} \hat{\rho}_{A'A}]$ in order to construct the associated \mathcal{A} map as follows:

$$\text{Tr}_{A'A}[\hat{\Pi}_i^{A'A} \hat{\rho}_{A'A}] = \text{Tr}_{A'A}[\hat{\Pi}_i^{A'A} \mathcal{A}(\hat{\rho}_A)] = \mathcal{T} \circ \Pi \circ \tilde{\mathcal{A}}(\hat{\rho}_A), \quad (35)$$

where $\tilde{\mathcal{A}}(\hat{\rho}_A) = \hat{\rho}_{A'A}$ defines the *assignment map*²³. The assignment map is linear i.e.,

$$\tilde{\mathcal{A}}(P_\alpha^A) = \tau_\alpha^{A'} \otimes P_\alpha^A \quad (36)$$

$$\Rightarrow \tilde{\mathcal{A}}\left(\sum_k r_k P_k^A\right) = \sum_\alpha r_\alpha \tau_\alpha^{A'} \otimes P_\alpha^A \quad (37)$$

where P_α^A are linearly independent states of system A . Let $\{Q_\beta\}$ be a set of hermitian operators such that

$$\begin{aligned} \text{Tr}[P_\alpha^A Q_\beta] &= \delta_{\alpha,\beta} \\ \sum_\beta Q_\beta &= I_A. \end{aligned} \quad (38)$$

We can thus express

$$\tilde{\mathcal{A}} = \sum_\alpha \tau_\alpha^{A'} \otimes P_\alpha^A \otimes Q_\alpha^T. \quad (39)$$

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(With the above construction, it may be readily identified that

$$\begin{aligned}
 \left[\tilde{\mathcal{A}} \left(\sum_{\beta} r_{\beta} P_{\beta}^A \right) \right]_{a'_i a_i; a'_j a_j} &= \sum_{\alpha} \sum_{a_k, a_l} \left[\tau_{\alpha}^{A'} \otimes P_{\alpha}^A \otimes Q_{\alpha}^T \right]_{a'_i a_i a_k; a'_j a_j a_l} \left[\sum_{\beta} r_{\beta} P_{\beta}^A \right]_{a_k a_l} \\
 &= \sum_{\alpha, \beta} r_{\beta} \sum_{a_k, a_l} \left[\tau_{\alpha}^{A'} \otimes P_{\alpha}^A \right]_{a'_i a_i; a'_j a_j} [Q_{\alpha}^T]_{a_k a_l} [P_{\beta}^A]_{a_k a_l} \\
 &= \sum_{\alpha, \beta} r_{\beta} \left[\tau_{\alpha}^{A'} \otimes P_{\alpha}^A \right]_{a'_i a_i; a'_j a_j} \text{Tr}[P_{\beta}^A Q_{\alpha}] \\
 &= \sum_{\alpha, \beta} r_{\beta} \left[\tau_{\alpha}^{A'} \otimes P_{\alpha}^A \right]_{a'_i a_i; a'_j a_j} \delta_{\alpha, \beta} \\
 &= \sum_{\alpha} r_{\alpha} \left[\tau_{\alpha}^{A'} \otimes P_{\alpha}^A \right]_{a'_i a_i; a'_j a_j}
 \end{aligned}$$

or we obtain, $\tilde{\mathcal{A}}(\sum_{\alpha} r_{\alpha} P_{\alpha}^A) = \sum_{\alpha} r_{\alpha} \tau_{\alpha}^{A'} \otimes P_{\alpha}^A$ as expected). Substituting (39) in (35) we obtain,

$$\begin{aligned}
 \mathcal{P}_i &= \text{Tr}_{A'A} [\hat{\Pi}_i^{A'A} \hat{\rho}_{A'A}] = \text{Tr}_{A'A} [\hat{\Pi}_i^{A'A} \tilde{\mathcal{A}}(\hat{\rho}_A)] \\
 &= \sum_{a'_i, a_j, a'_k, a_l, a_s, a_t} \left[\hat{\Pi}_i^{A'A} \right]_{a'_i, a_j; a_l, a_s} \left[\tilde{\mathcal{A}} \right]_{a'_i a_j a_s; a'_k a_l a_t} [(\hat{\rho}_A)]_{a_s a_t}
 \end{aligned} \tag{40}$$

and substituting (40) back in (32), we identify the following:

$$\begin{aligned}
 [\hat{\rho}_A^{\mathcal{R}}]_{a_u a_v} &= \sum_i [\hat{\rho}_i^A]_{a_u a_v} \left\{ \sum_{a'_i, a_j, a'_k, a_l, a_s, a_t} \left[\hat{\Pi}_i^{A'A} \right]_{a'_i, a_j; a_l, a_s} \left[\tilde{\mathcal{A}} \right]_{a'_i a_j a_s; a'_k a_l a_t} [(\hat{\rho}_A)]_{a_s a_t} \right\} \\
 &= \sum_{a_s, a_t} \mathcal{A}_{a_u a_v; a_s a_t} [(\hat{\rho}_A)]_{a_s a_t}
 \end{aligned} \tag{41}$$

where we identify the elements of the \mathcal{A} -matrix transforming the initial state ρ_A to final state $\rho_A^{\mathcal{R}}$ as,

$$\begin{aligned}
 \mathcal{A}_{a_u a_v; a_s a_t} &= \sum_i [\hat{\rho}_i^A]_{a_s a_t} \left\{ \sum_{a'_i, a_j, a_l, a_s} \left[\hat{\Pi}_i^{A'A} \right]_{a'_i, a_j; a_l, a_s} \tilde{\mathcal{A}}_{a'_i a_j a_s; a'_k a_l a_t} \right\} \\
 &= \sum_i \sum_{a'_i, a_j, a_l, a_s} [\hat{\rho}_i^A]_{a_s a_t} \left[\hat{\Pi}_i^{A'A} \right]_{a'_i, a_j; a_l, a_s} \tilde{\mathcal{A}}_{a'_i a_j a_s; a'_k a_l a_t}
 \end{aligned} \tag{42}$$

The elements of the corresponding realigned \mathcal{B} matrix (see 27)) are then identified

as,

$$\begin{aligned}
\mathcal{B}_{a_u a_s; a_v a_t} &= \mathcal{A}_{a_u a_v; a_s a_t} \\
&= \sum_i \sum_{a'_i, a_j, a_l, a_s} [\hat{\rho}_i^A]_{a_s a_t} [\hat{\Pi}_i^{A'A}]_{a'_i, a_j; a_l, a_s} \tilde{\mathcal{A}}_{a'_i a_j a_s; a'_k a_l a_t} \\
\Rightarrow \mathcal{B} &= \sum_i \hat{\rho}_i^A \otimes \text{Tr}_{A'A} [\hat{\Pi}_i^{A'A} \tilde{\mathcal{A}}] \\
&= \sum_i \sum_{\alpha} \hat{\rho}_i^A \otimes \text{Tr}_{A'A} [\hat{\Pi}_i^{A'A} (\tau_{\alpha}^{A'} \otimes P_{\alpha}^A \otimes Q_{\alpha}^T)] \\
&= \sum_i \sum_{\alpha} \text{Tr}_{A'A} [\hat{\Pi}_i^{A'A} (\tau_{\alpha}^{A'} \otimes P_{\alpha}^A)] \rho_i^A \otimes Q_{\alpha}^T \\
\mathcal{B} &= \sum_{\alpha} \left[\sum_i q_{i\alpha} \hat{\rho}_i^A \right] \otimes Q_{\alpha}^T \\
&= \sum_{\alpha} \hat{\eta}_{\alpha}^A \otimes Q_{\alpha}^T
\end{aligned} \tag{43}$$

where we have denoted

$$\begin{aligned}
\sum_{\alpha} q_{i\alpha} \hat{\rho}_i^A &= \hat{\eta}_{\alpha}^A, \\
q_{i\alpha} &= \text{Tr}_{A'A} [\hat{\Pi}_i^{A'A} (\tau_{\alpha}^{A'} \otimes P_{\alpha}^A)].
\end{aligned} \tag{44}$$

Now, we consider a specific example of two qubit state (see (21))

$$\begin{aligned}
\hat{\rho}^{A'A} &= \text{Tr}_B [\hat{\rho}_{A'AB}] \\
&= p |1_{A'}, 0_A\rangle \langle 1_{A'}, 0_A| + (1-p) |0_{A'}, +_A\rangle \langle 0_{A'}, +_A|
\end{aligned} \tag{45}$$

and complete, orthogonal projective measurement (22).

We choose the following set $\{P_{\alpha}^A\}$ of linearly independent 2×2 matrices (see (36)), which serve as a basis²³ for single qubit systems:

$$\begin{aligned}
P_1^A &= \frac{1}{2}[I + \sigma_1] = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_2^A = \frac{1}{2}[I + \sigma_2] = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \\
P_3^A &= \frac{1}{2}[I + \sigma_3] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_4^A = \frac{1}{2}[I - \sigma_1] = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\end{aligned} \tag{46}$$

The corresponding set of Hermitian matrices $\{Q_{\beta}\}$, which are orthogonal to $\{P_{\alpha}^A\}$ and obey the property $\sum_{\beta} Q_{\beta} = I$ (see Eq. (38)) are given by,

$$\begin{aligned}
Q_1 &= \frac{1}{2}(I + \sigma_1 + \sigma_2 - \sigma_3) = \frac{1}{2} \begin{pmatrix} 0 & 1-i \\ 1+i & 2 \end{pmatrix}, \quad Q_2 = -\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
Q_3 &= \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_4 = \frac{1}{2}(I - \sigma_1 + \sigma_2 - \sigma_3) = \frac{1}{2} \begin{pmatrix} 0 & -1-i \\ -1+i & 2 \end{pmatrix}
\end{aligned} \tag{47}$$

Further, choosing

$$\tau_{1,4}^{A'} = |0\rangle_{A'} \langle 0|, \quad \tau_{2,3}^{A'} = |1\rangle_{A'} \langle 1| \tag{48}$$

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in (39) and simplifying (using (45), (46) and (47) we obtain

$$\tilde{\mathcal{A}}(\rho^A) = \tilde{\mathcal{A}}((1-p)P_1^A + pP_3^A) = (1-p)P_1 \otimes \tau_1 + pP_3 \otimes \tau_3 = \hat{\rho}_{A'A} \quad (49)$$

confirming the consistency of the assignment map $\tilde{\mathcal{A}}$.

Using the explicit matrices $\{P_\alpha^A\}$, $\{Q_\beta\}$ of (46), (47), and (48), along with (22) for projective measurements, we obtain (see (45))

$$\begin{aligned} q_{i1} &= \text{Tr}_{A'A}[\hat{\Pi}_i^{A'A} (P_1^A \otimes \tau_1)] = (1, 0, 0, 0) \\ q_{i2} &= \text{Tr}_{A'A}[\hat{\Pi}_i^{A'A} (P_2^A \otimes \tau_2)] = (0, 0, \frac{1}{2}, \frac{1}{2}) \\ q_{i3} &= \text{Tr}_{A'A}[\hat{\Pi}_i^{A'A} (P_3^A \otimes \tau_3)] = (0, 0, 1, 0) \\ q_{i4} &= \text{Tr}_{A'A}[\hat{\Pi}_i^{A'A} (P_4^A \otimes \tau_4)] = (0, 1, 0, 0) \end{aligned} \quad (50)$$

and (see ((33), 44))

$$\begin{aligned} \eta_1 &= \sum_i q_{i1} \rho_i^A = \sum_i q_{i1} \text{Tr}_E[\hat{\Pi}_i^{A'A}] = \text{Tr}_E[\hat{\Pi}_1^{A'A}] = |+\rangle\langle+| = P_1 \\ \eta_2 &= \sum_i q_{i2} \rho_i^A = \sum_i q_{i2} \text{Tr}_E[\hat{\Pi}_i^{A'A}] = \frac{1}{2} (\text{Tr}_E[\hat{\Pi}_3^{A'A}] + \text{Tr}_E[\hat{\Pi}_4^{A'A}]) \\ &= \frac{1}{2} (|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I}{2} \\ \eta_3 &= \sum_i q_{i3} \rho_i^A = \sum_i q_{i3} \text{Tr}_E[\hat{\Pi}_i^{A'A}] = \text{Tr}_E[\hat{\Pi}_3^{A'A}] = |0\rangle\langle 0| = P_3 \\ \eta_4 &= \sum_i q_{i4} \rho_i^A = \sum_i q_{i4} \text{Tr}_E[\hat{\Pi}_i^{A'A}] = \text{Tr}_E[\hat{\Pi}_2^{A'A}] = |-\rangle\langle-| = P_4 \end{aligned} \quad (51)$$

We thus obtain the \mathcal{B} map (see (43)) corresponding to this particular example as,

$$\begin{aligned} \mathcal{B} &= P_1 \otimes Q_1 + \frac{I}{2} \otimes Q_2 + P_3 \otimes Q_3 + P_4 \otimes Q_4 \\ &= \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (52)$$

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